DAY TWENTY

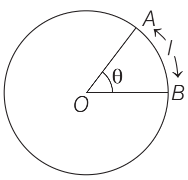
Trigonometric Functions and Equations

Learning & Revision for the Day

- Angle on Circular System
- Trigonometric Functions
- Trigonometric Identities
- Trigonometric Ratios/Functions of Acute Angles
- Trigonometric Ratios of Compound and Multiple Angles
- Transformation Formulae
- Conditional Identities
- Maximum and Minimum Values
- Trigonometric Equations
- Summation of Some Trigonometric Series

Angle on Circular System

If the angle subtended by an arc of length l at the centre of a circle of radius r is θ, then $\theta = \frac{l}{r}$

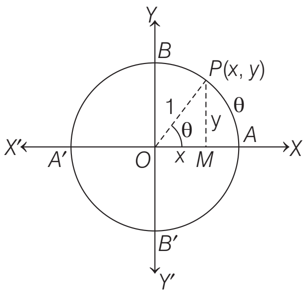


If the length of arc is equal to the radius of the circle, then the angle subtended at the centre of the circle will be one radian. One radian is denoted by 1^c and $1^c = 57°16'22''$ approximately.

(Figure shows the angle whose measure are one radian.)

$$2\pi^C = 360°, 1^C = \frac{180°}{\pi}, 1° = \left(\frac{\pi}{180}\right)^C$$

Trigonometric Functions



Let $X'OX$ and YOY' be the coordinate axes. Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B' as shown in the figure

Also, let $P(x, y)$ be any on the circle with $\angle AOP = \theta$ radian, i.e. length of arc $AP = \theta$

Then, the six trigonometric functions, can be defined as

(i) $\cos\theta = \dfrac{OM}{OP} = x$

(ii) $\sin\theta = \dfrac{PM}{OP} = y$

(iii) $\sec\theta = \dfrac{OP}{OM} = \dfrac{1}{x}, x \neq 0$

(iv) $\text{cosec}\,\theta = \dfrac{OP}{PM} = \dfrac{1}{y}, y \neq 0$

(v) $\tan\theta = \dfrac{PM}{OM} = \dfrac{y}{x}, x \neq 0$

(vi) $\cot\theta = \dfrac{OM}{PM} = \dfrac{x}{y}, y \neq 0$

Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions which is true for all those angles for which the functions are defined is called trigonometric identity.

Some identities are given below

(i) $\sin^2\theta + \cos^2\theta = 1$

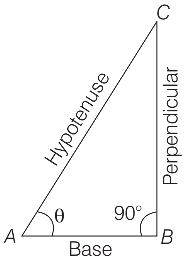
(ii) $\sec^2\theta - \tan^2\theta = 1$

(iii) $\text{cosec}^2\theta - \cot^2\theta = 1$

Trigonometric Ratios/Functions of Acute Angles

The ratios of the sides of a triangles with respect to its acute angles are called trigonometric ratios or T-ratios.

In a right angled ΔABC, if $\angle CAB = \theta$, then



1. $\sin\theta = \dfrac{BC}{AC} = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$
2. $\cos\theta = \dfrac{AB}{AC} = \dfrac{\text{Base}}{\text{Hypotenuse}}$
3. $\tan\theta = \dfrac{BC}{AB} = \dfrac{\text{Perpendicular}}{\text{Base}}$
4. $\text{cosec}\,\theta = \dfrac{1}{\sin\theta} = \dfrac{AC}{BC} = \dfrac{\text{Hypotenuse}}{\text{Perpendicular}}$
5. $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{AC}{AB} = \dfrac{\text{Hypotenuse}}{\text{Base}}$
6. $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{AB}{BC} = \dfrac{\text{Base}}{\text{Perpendicular}}$

Sign for Trigonometric Ratios in four Quadrants

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
I. $(0, 90°)$	+	+	+	+	+	+
II. $(90°, 180°)$	+	–	–	–	–	+
III. $(180°, 270°)$	–	–	+	+	–	–
IV. $(270°, 360°)$	–	+	–	–	+	–

Trigonometric ratios of some useful angles between 0°and 90°

Angle	$0°/0$	$15°/\frac{\pi}{12}$	$18°/\frac{\pi}{10}$	$22.5°/\frac{\pi}{8}$	$30°/\frac{\pi}{6}$	$36°/\frac{\pi}{5}$	$45°/\frac{\pi}{4}$	$54°/\frac{3\pi}{10}$	$60°/\frac{\pi}{3}$	$67.5°/\frac{3\pi}{8}$	$72°/\frac{2\pi}{5}$	$75°/\frac{5\pi}{12}$	$90°/\frac{\pi}{2}$
$\sin\theta$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1
$\cos\theta$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0
$\tan\theta$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	1	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{3}$	$\sqrt{2}+1$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞
$\cot\theta$	∞	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\sqrt{2}+1$	$\sqrt{3}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	1	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	0
$\sec\theta$	1	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}-1$	$\sqrt{2}$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	2	$\sqrt{4+2\sqrt{2}}$	$\sqrt{5}+1$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	∞
$\text{cosec}\,\theta$	∞	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	$\sqrt{5}+1$	$\sqrt{4+2\sqrt{2}}$	2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{2}$	$\sqrt{5}-1$	$\frac{2}{\sqrt{3}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	1

Trigonometric ratios of allied angles

θ	sin θ	cosec θ	cos θ	sec θ	tan θ	cot θ
$-\theta$	$-\sin\theta$	$-\text{cosec}\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$
$90^\circ-\theta$	$\cos\theta$	$\sec\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$\cot\theta$	$\tan\theta$
$90^\circ+\theta$	$\cos\theta$	$\sec\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$-\cot\theta$	$-\tan\theta$
$180^\circ-\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$-\cos\theta$	$-\sec\theta$	$-\tan\theta$	$-\cot\theta$
$180^\circ+\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$-\cos\theta$	$-\sec\theta$	$\tan\theta$	$\cot\theta$
$270^\circ-\theta$	$-\cos\theta$	$-\sec\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$\cot\theta$	$\tan\theta$
$270^\circ+\theta$	$-\cos\theta$	$-\sec\theta$	$\sin\theta$	$\text{cosec}\,\theta$	$-\cot\theta$	$-\tan\theta$
$360^\circ-\theta$	$-\sin\theta$	$-\text{cosec}\,\theta$	$\cos\theta$	$\sec\theta$	$-\tan\theta$	$-\cot\theta$

Trigonometric Ratios of Compound and Multiple Angles

Compound Angles

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(iii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(iv) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(v) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(vii) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

(viii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(ix) (a) $\dfrac{1+\tan A}{1-\tan A}=\tan\left(\dfrac{\pi}{4}+A\right)$

(b) $\dfrac{1-\tan A}{1+\tan A}=\tan\left(\dfrac{\pi}{4}-A\right)$

Multiple Angles

(i) $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A=\cos^2 A-\sin^2 A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$=2\cos^2 A-1=1-2\sin^2 A$

(iii) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

(iv) $\sin 3A=3\sin A-4\sin^3 A$

(v) $\cos 3A=4\cos^3 A-3\cos A$

(vi) $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

Transformation Formulae

(i) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(ii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(iii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(iv) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

(v) $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(vi) $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(vii) $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(viii) $\cos C-\cos D=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(ix) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

(x) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$

(xi) $\cos A\cos 2A\cos 4A\cos 8A\ldots\cos 2^{n-1}A$

$=\dfrac{1}{2^n\sin A}\sin(2^n A)$

Conditional Identities

If $A+B+C=180^\circ$, then

(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

(ii) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(iii) $\sin A+\sin B+\sin C=4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iv) $\cos A+\cos B+\cos C=1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(v) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(vi) $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$

Maximum and Minimum Values

1. $-1\le\sin x\le 1, |\sin x|\le 1$
2. $-1\le\cos x\le 1, |\cos x|\le 1$
3. $|\sec x|\ge 1, |\text{cosec}\, x|\ge 1$
4. $\tan x, \cot x$ take all real values

NOTE
- Maximum value of $a\cos\theta \pm b\sin\theta = \sqrt{a^2+b^2}$
- Minimum value of $a\cos\theta \pm b\sin\theta = -\sqrt{a^2+b^2}$
- Maximum value of $a\cos\theta \pm b\sin\theta + c = c + \sqrt{a^2+b^2}$
- Minimum value of $a\cos\theta \pm b\sin\theta + c = c - \sqrt{a^2+b^2}$

Trigonometric Equations

An equation involving one or more trigonometrical ratios of unknown angle is called a **trigonometrical equation**.

e.g. $\sin\theta + \cos^2\theta = 0$

Principal Solution

The value of the unknown angle (say θ) which satisfies the trigonometric equation is known as principal solution, if $0 \le \theta < 2\pi$.

General Solution

Since, trigonometrical functions are periodic function, solution of trigonometric equation can be generalised with the help of the periodicity of the trigonometrical functions.

The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Some general trigonometric equations and their solutions

Equations	Solutions	Equations	Solutions
$\sin x = \sin\alpha\left(-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}\right)$	$x = n\pi + (-1)^n\alpha$, $n \in I$	$\sin^2 x = \sin^2\alpha$	$x = n\pi \pm \alpha,\ n \in I$
$\cos x = \cos\alpha\ (0 \le \alpha \le \pi)$	$x = 2n\pi \pm \alpha$, $n \in I$	$\cos^2 x = \cos^2\alpha$	
$\tan x = \tan\alpha\left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right)$	$x = n\pi + \alpha,\ n \in I$	$\tan^2 x = \tan^2\alpha$	
$\sin x = 0$	$x = n\pi,\ n \in I$	$\sin x = 1$	$x = (4n+1)\pi/2,\ n \in I$
$\cos x = 0$	$x = (2n+1)\pi/2,\ n \in I$	$\cos x = 1$	$x = 2n\pi,\ n \in I$
$\tan x = 0$	$x = n\pi,\ n \in I$	$\cos x = -1$	$x = (2n+1)\pi,\ n \in I$
		$\sin x = \sin\alpha$ and $\cos x = \cos\alpha$	$x = 2n\pi + \alpha,\ n \in I$

Summation of Some Trigonometric Series

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots$ to n terms

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \sin\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots$ to n terms

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \cos\left\{\alpha + (n-1)\left(\frac{\beta}{2}\right)\right\}$$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $\dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}=x$, then $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is equal to

(a) x (b) $\dfrac{1}{x}$ (c) $1-x$ (d) $1+x$

2 If $\dfrac{3\pi}{4}<\alpha<\pi$, then $\sqrt{\operatorname{cosec}^2\alpha+2\cot\alpha}$ is equal to

(a) $1+\cot\alpha$ (b) $1-\cot\alpha$
(c) $-1-\cot\alpha$ (d) $-1+\cot\alpha$

3 The least value of $\operatorname{cosec}^2 x+25\sec^2 x$ is

(a) 0 (b) 26 (c) 28 (d) 36

4 If $\tan A+\sin A=m$ and $\tan A-\sin A=n$, then $\dfrac{(m^2-n^2)^2}{mn}$ is equal to

(a) 4 (b) 3 (c) 16 (d) 9

5 If $a\cos^3\alpha+3a\cos\alpha\sin^2\alpha=m$ and $a\sin^3\alpha+3a\cos^2\alpha\sin\alpha=n$, then $(m+n)^{2/3}+(m-n)^{2/3}$ is equal to

(a) $2a^2$ (b) $2a^{1/3}$ (c) $2a^{2/3}$ (d) $2a^3$

6 $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$ is equal to

(a) 1 (b) $\cos\dfrac{\pi}{8}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$

7 If $\sin\theta=-\dfrac{4}{5}$ and θ lies in the third quadrant, then $\cos\dfrac{\theta}{2}$ is equal to → **NCERT Exemplar**

(a) $\dfrac{1}{\sqrt{5}}$ (b) $-\dfrac{1}{\sqrt{5}}$ (c) $\sqrt{\dfrac{2}{5}}$ (d) $-\sqrt{\dfrac{2}{5}}$

8 The expression $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$ can be written as → **JEE Mains 2013**

(a) $\sin A\cos A+1$ (b) $\sec A\operatorname{cosec} A+1$
(c) $\tan A+\cot A$ (d) $\sec A+\operatorname{cosec} A$

9 If $\tan\theta$, $2\tan\theta+2$ and $3\tan\theta+3$ are in GP, then the value of $\dfrac{7-5\cot\theta}{9-4\sqrt{\sec^2\theta-1}}$ is

(a) $\dfrac{12}{5}$ (b) $-\dfrac{33}{28}$ (c) $\dfrac{33}{100}$ (d) $\dfrac{12}{13}$

10 If $\sin(\alpha+\beta)=1$ and $\sin(\alpha-\beta)=\dfrac{1}{2}$, then $\tan(\alpha+2\beta)\tan(2\alpha+\beta)$ is equal to

(a) 1 (b) -1
(c) zero (d) None of these

11 In an acute angled triangle, the least value of $\sec A+\sec B+\sec C$ is

(a) 3 (b) 4 (c) 5 (d) 6

12 If $0<A<B<\pi$, $\sin A+\sin B=\sqrt{\dfrac{3}{2}}$ and $\cos A+\cos B=\dfrac{1}{\sqrt{2}}$, then A is equal to

(a) $15°$ (b) $30°$ (c) $45°$ (d) $22\dfrac{1}{2}°$

13 In a ΔPQR, $\angle R=\dfrac{\pi}{2}$. If $\tan\left(\dfrac{P}{2}\right)$ and $\tan\left(\dfrac{Q}{2}\right)$ are the roots of $ax^2+bx+c=0$, where $a\neq 0$, then

(a) $b=a+c$ (b) $b=c$
(c) $c=a+b$ (d) $a=b+c$

14 If $\cos(\alpha+\beta)=\dfrac{4}{5}$ and $\sin(\alpha-\beta)=\dfrac{5}{13}$, where $0\le\alpha,\ \beta\le\dfrac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to → **AIEEE 2010**

(a) $\dfrac{25}{16}$ (b) $\dfrac{56}{33}$ (c) $\dfrac{19}{12}$ (d) $\dfrac{20}{7}$

15 If $\sin\alpha=x$, $\sin\beta=y$, $\sin(\alpha+\beta)=z$, then $\cos(\alpha+\beta)$ as a rational function is

(a) $\dfrac{z^2-x^2-y^2}{xy}$ (b) $\dfrac{z^2-x^2-y^2}{2xy}$
(c) $\dfrac{z^2+x^2+y^2}{xy}$ (d) $\dfrac{z^2+x^2+y^2}{2xy}$

16 Let α and β be such that $\pi<\alpha-\beta<3\pi$. If $\sin\alpha+\sin\beta=-\dfrac{21}{65}$ and $\cos\alpha+\cos\beta=-\dfrac{27}{65}$, then the value of $\cos\left(\dfrac{\alpha-\beta}{2}\right)$ is → **AIEEE 2004**

(a) $-\dfrac{3}{\sqrt{130}}$ (b) $\dfrac{3}{\sqrt{130}}$ (c) $\dfrac{6}{65}$ (d) $-\dfrac{6}{65}$

17 If $A+B+C=\pi$ and $\cos A=\cos B\cos C$, then $\tan B\tan C$ is equal to

(a) $\dfrac{1}{2}$ (b) 2 (c) 1 (d) $-\dfrac{1}{2}$

18 If $\tan\alpha=(1+2^{-x})^{-1}$, $\tan\beta=(1+2^{x+1})^{-1}$, then $\alpha+\beta$ equals

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

19 If $0<x<\pi$ and $\cos x+\sin x=\dfrac{1}{2}$, then $\tan x$ is equal to

(a) $\dfrac{-(4+\sqrt{7})}{3}$ (b) $\dfrac{1+\sqrt{7}}{4}$ (c) $\dfrac{1-\sqrt{7}}{4}$ (d) $\dfrac{4-\sqrt{7}}{3}$

20 $\operatorname{cosec}10°-\sqrt{3}\sec 10°$ is equal to

(a) 1 (b) 1/2 (c) 2 (d) 4

21 The value of $\sin 12°\sin 48°\sin 54°$ is equal to

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{32}$ (c) $\dfrac{1}{8}$ (d) $\dfrac{1}{4}$

22 The value of $1 + \cos 56° + \cos 58° - \cos 66°$ is equal to

(a) $2\cos 28° \cos 29° \cos 33°$ (b) $4\cos 28° \cos 29° \sin 33°$
(c) $4\cos 28° \cos 29° \cos 33°$ (d) $2\cos 28° \cos 29° \sin 33°$

23 Let n be a positive integer such that

$\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then,

(a) $n = 6$ (b) $n = 1, 2, 3, \ldots, 8$
(c) $n = 5$ (d) None of these

24 The value of $\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A$ is

(a) $\cot A$ (b) $\tan A$ (c) $\cos A$ (d) $\sin A$

25 The maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ is attained at

(a) $x = \frac{\pi}{12}$ (b) $x = \frac{\pi}{6}$ (c) $x = \frac{\pi}{3}$ (d) $x = \frac{\pi}{2}$

26 If $\cos A = m\cos B$ and $\cot \frac{A+B}{2} = \lambda \tan \frac{B-A}{2}$, then λ is

(a) $\frac{m}{m-1}$ (b) $\frac{m+1}{m}$
(c) $\frac{m+1}{m-1}$ (d) None of these

27 If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$, then the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is equal to

(a) 1 (b) 2 (c) 0 (d) $3\cos\theta$

28 The maximum value of $\cos^2\left(\frac{\pi}{3} - x\right) - \cos^2\left(\frac{\pi}{3} + x\right)$ is

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{2}$

29 If sum of all the solution of the equation

$$8\cos x \cdot \left(\cos\left(\frac{\pi}{6} + x\right) \cdot \cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2}\right)$$

$= 1$ in $[0, \pi]$ is $k\pi$, then k is equal to

(a) $\frac{2}{3}$ (b) $\frac{13}{9}$ (c) $\frac{8}{9}$ (d) $\frac{20}{9}$

30 $\sin^2\theta = \frac{4xy}{(x+y)^2}$ is true, if and only if → **AIEEE 2002**

(a) $x - y \neq 0$ (b) $x = -y$
(c) $x + y \neq 0$ (d) $x \neq 0, y \neq 0$

31 The smallest positive integral value of p for which the equation $\cos(p\sin x) = \sin(p\cos x)$ in x has a solution in $[0, 2\pi]$ is

(a) 2 (b) 1 (c) 3 (d) 5

32 If $A = \sin^2 x + \cos^4 x$, then for all real x

(a) $\frac{13}{16} \le A \le 1$ (b) $1 \le A \le 2$
(c) $\frac{3}{4} \le A \le \frac{13}{16}$ (d) $\frac{3}{4} \le A \le 1$

33 $\cos 2\theta + 2\cos\theta$ is always

(a) greater than $-\frac{3}{2}$ (b) less than or equal to $\frac{3}{2}$
(c) greater than or equal to $-\frac{3}{2}$ and less than or equal to 3
(d) None of the above

34 If $f: R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is

(a) $[0, 3]$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) $[-1, 3]$

35 If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is

(a) π (b) 2π (c) $\frac{5\pi}{2}$ (d) None of these

36 The number of solution of $\cos x = |1 + \sin x|$, $0 \le x \le 3\pi$ is

(a) 2 (b) 3 (c) 4 (d) 5

37 If $\sin 2x + \cos x = 0$, then which among the following is/are true?

I. $\cos x = 0$ II. $\sin x = -\frac{1}{2}$

III. $x = (2n+1)\frac{\pi}{2}, n \in Z$ IV. $x = n\pi + (-1)^n \frac{7\pi}{6}, n \in Z$

(a) I is true (b) I and II are true
(c) I, II and III are true (d) All are true

38 The number of solutions of the equation $\sin 2x - 2\cos x + 4\sin x = 4$ in the interval $[0, 5\pi]$ is → **JEE Mains 2013**

(a) 3 (b) 5 (c) 4 (d) 6

39 The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are

(a) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (b) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$ (d) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

40 The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is

(a) 6 (b) 1 (c) 2 (d) 4

41 If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to

(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$ (c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

42 **Statement I** $\cos\frac{\pi}{7} \cdot \cos\frac{2\pi}{7} \cdot \cos\frac{4\pi}{7} = -1/8$

Statement II $\cos\theta \cos 2\theta \cos 4\theta \ldots$

$$\cos 2^{n-1}\theta = -\frac{1}{2^n} \text{ for } \theta = \frac{\pi}{2^n - 1}$$

(a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where $x \in R, k \geq 1$, then $f_4(x) - f_6(x)$ is equal to → **JEE Mains 2014**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

2 If $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ are the roots of the equation $x^3 - px^2 - r = 0$, then the value of $(1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)$ is equal to

(a) $(p-r)^2$ (b) $1+(p-r)^2$
(c) $1-(p-r)^2$ (d) None of these

3 For $0 < \phi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xy - z$
(c) $xyz = x + y + z$ (d) $xyz = yz + x$

4 If $\sin^4 x + \cos^4 y + 2 = 4\sin x \cos y$ and $0 \leq x, y \leq \frac{\pi}{2}$, then $\sin x + \cos y$ is equal to

(a) –2 (b) 0 (c) 2 (d) 3/2

5 If $0 \leq x < 2\pi$, then the number of real values of x, which satisfy the equation

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0, \text{ is}$$

(a) 3 (b) 5
(c) 7 (d) 9

6 If $\sin(\theta+\alpha) = a$ and $\sin(\theta+\beta) = b$, then $\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta)$ is equal to

(a) $1-a^2-b^2$ (b) $1-2a^2-2b^2$
(c) $2+a^2-b^2$ (d) $2-a^2-b^2$

7 If $\alpha = \sin^8\theta + \cos^{14}\theta$, then which of the following is true?

(a) $\alpha > 1$ *(b)* $\alpha \leq 1$ *(c)* $\alpha = 0$ *(d)* $\alpha < 0$

8 Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r\theta$ for all real θ, then

(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$
(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - 3n - 3$

9 If $x\sin a + y\sin 2a + z\sin 3a = \sin 4a$

$x\sin b + y\sin 2b + z\sin 3b = \sin 4b$

$x\sin c + y\sin 2c + z\sin 3c = \sin 4c$

Then, the roots of the equation

$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0$, $a, b, c \neq n\pi$, are

(a) $\sin a, \sin b, \sin c$ (b) $\cos a, \cos b, \cos c$
(c) $\sin 2a, \sin 2b, \sin 2c$ (d) $\cos 2a, \cos 2b, \cos 2c$

10 The number of solutions of the equation $|\sin\theta \cdot \cos\theta| + \sqrt{2+\tan^2\theta+\cot^2\theta} = \sqrt{3}, \theta \in [0, 4\pi]$ is/are

(a) 0 (b) 1 (c) 2 (d) 3

11 The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(\sqrt{3})^{\sec^2\theta} = \tan^4\theta + 2\tan^2\theta$ is

(a) 1 (b) 2 (c) 3 (d) None of these

12 The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has **AIEEE 2012**

(a) infinite number of real roots
(b) no real roots
(c) exactly one real root
(d) exactly four real roots

13 Find the general solution of the equation $(\sqrt{3}-1)\cos\theta + (\sqrt{3}+1)\sin\theta = 2$.

(a) $2n\pi \pm \frac{\pi}{4} - \frac{5\pi}{12}$ (b) $2n\pi \pm \frac{\pi}{4} + \frac{5\pi}{12}$
(c) $2n\pi \pm \pi - \frac{3\pi}{12}$ (d) None of these

14 At how many points the curve $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$ will intersect X-axis in the region $-\pi \leq x \leq \pi$?

(a) 4 (b) 6 (c) 8 (d) None of these

15 In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to → **AIEEE 2012**

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

ANSWERS

SESSION 1	**1.** (a)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (a)
	11. (d)	**12.** (a)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (a)	**17.** (b)	**18.** (b)	**19.** (a)	**20.** (d)
	21. (c)	**22.** (b)	**23.** (a)	**24.** (a)	**25.** (a)	**26.** (c)	**27.** (c)	**28.** (c)	**29.** (b)	**30.** (c)
	31. (a)	**32.** (d)	**33.** (c)	**34.** (d)	**35.** (b)	**36.** (b)	**37.** (d)	**38.** (a)	**39.** (a)	**40.** (d)
	41. (b)	**42.** (a)								

SESSION 2	**1.** (d)	**2.** (b)	**3.** (c)	**4.** (c)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (b)	**9.** (b)	**10.** (a)
	11. (b)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (b)					

Hints and Explanations

SESSION 1

1 Given, $\dfrac{2\sin\theta}{1+\cos\theta+\sin\theta} = x$

$\therefore\; x = \dfrac{2\sin\theta\,(1+\sin\theta-\cos\theta)}{(1+\sin\theta)^2-\cos^2\theta}$

$= \dfrac{2\sin\theta\,(1+\sin\theta-\cos\theta)}{1^2+\sin^2\theta+2\sin\theta-(1-\sin^2\theta)}$

$= \dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta}$

2 We have, $\sqrt{\operatorname{cosec}^2\alpha + 2\cot\alpha}$

$= \sqrt{1+\cot^2\alpha+2\cot\alpha}$

$= |1+\cot\alpha|$

But $\dfrac{3\pi}{4} < \alpha < \pi$

$\Rightarrow\quad \cot\alpha < -1$

$\Rightarrow\quad 1+\cot\alpha < 0$

Hence, $|1+\cot\alpha| = -(1+\cot\alpha)$

3 $\operatorname{cosec}^2 x + 25\sec^2 x$

$= 1+\cot^2 x + 25(1+\tan^2 x)$

$= 26+\cot^2 x + 25\tan^2 x$

$= 26+10+(\cot x - 5\tan x)^2 \geq 36$

4 Since, $\tan A + \sin A = m$

and $\tan A - \sin A = n$

$\therefore\quad m+n = 2\tan A$

and $m-n = 2\sin A$

Also, $mn = (\tan A+\sin A)(\tan A - \sin A)$

$= \tan^2 A - \sin^2 A$

Now, $\dfrac{(m^2-n^2)^2}{mn} = \dfrac{(m+n)^2(m-n)^2}{mn}$

$= \dfrac{(2\tan A)^2(2\sin A)^2}{\tan^2 A-\sin^2 A}$

$= \dfrac{16\tan^2 A\sin^2 A}{\sin^2 A\tan^2 A} = 16$

5 Given,

$a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$

and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$

$\therefore (m+n) = a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha + 3a\cos^2\alpha\sin\alpha + a\sin^3\alpha$

$= a(\cos\alpha+\sin\alpha)^3$

and similarly,

$(m-n) = a(\cos\alpha-\sin\alpha)^3$

$\therefore\quad (m+n)^{2/3}+(m-n)^{2/3}$

$= a^{2/3}\{(\cos\alpha+\sin\alpha)^2 + (\cos\alpha-\sin\alpha)^2\}$

$= a^{2/3}\{2(\cos^2\alpha+\sin^2\alpha)\}$

$= 2a^{2/3}$

6 Given expression

$= \left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{\pi}{8}\right)$

$= \left(1-\cos^2\dfrac{\pi}{8}\right)\left(1-\cos^2\dfrac{3\pi}{8}\right)$

$= \sin^2\dfrac{\pi}{8}\sin^2\dfrac{3\pi}{8} = \sin^2\dfrac{\pi}{8}\cos^2\dfrac{\pi}{8}$

$= \dfrac{1}{4}\sin^2\dfrac{\pi}{4} = \dfrac{1}{8}$

7 Given that, $\sin\theta = -\dfrac{4}{5}$ and θ lies in the 3rd quadrant.

$\therefore\quad \cos\theta = -\sqrt{1-\dfrac{16}{25}} = -\dfrac{3}{5}$

Now, $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

$= \pm\sqrt{\dfrac{1-\frac{3}{5}}{2}} = \pm\sqrt{\dfrac{1}{5}}$

But we take $\cos\dfrac{\theta}{2} = -\dfrac{1}{\sqrt5}$. Since, if θ lies in 3rd quadrant, then $\dfrac{\theta}{2}$ will be in 2nd quadrant.

Hence, $\cos\dfrac{\theta}{2} = -\dfrac{1}{\sqrt5}$

8 Given expression is

$\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A} = \dfrac{\sin A}{\cos A}\times\dfrac{\sin A}{\sin A-\cos A}+\dfrac{\cos A}{\sin A}\times\dfrac{\cos A}{\cos A-\sin A}$

$= \dfrac{1}{\sin A-\cos A}\left\{\dfrac{\sin^3 A-\cos^3 A}{\cos A\sin A}\right\}$

$= \dfrac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin A\cos A}$

$= \dfrac{1+\sin A\cos A}{\sin A\cos A}$

$= 1+\sec A\operatorname{cosec} A$

9 We have,

$(2\tan\theta+2)^2 = \tan\theta(3\tan\theta+3)$

$\Rightarrow\quad 4\tan^2\theta+8\tan\theta+4 = 3\tan^2\theta+3\tan\theta$

$\Rightarrow\quad \tan^2\theta+5\tan\theta+4 = 0$

$\Rightarrow\quad (\tan\theta+4)(\tan\theta+1) = 0$

$\Rightarrow\quad \tan\theta = -4\qquad (\because \tan\theta \neq -1)$

$\therefore \quad \dfrac{7-5\cot\theta}{9-4\sqrt{\tan^2\theta}} = \dfrac{7+\frac{5}{4}}{9-4(-4)} = \dfrac{33}{100}$

10 Since, $\sin(\alpha+\beta)=1$

$\therefore \quad \alpha+\beta = \dfrac{\pi}{2}$ ...(i)

and $\sin(\alpha-\beta) = \dfrac{1}{2}$

$\Rightarrow \quad \alpha-\beta = \dfrac{\pi}{6}$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\alpha = \dfrac{\pi}{3}$ and $\beta = \dfrac{\pi}{6}$

Now, $\tan(\alpha+2\beta)\tan(2\alpha+\beta)$

$= \tan\left(\dfrac{2\pi}{3}\right)\tan\left(\dfrac{5\pi}{6}\right)$

$= \tan\left(\pi-\dfrac{\pi}{3}\right)\tan\left(\pi-\dfrac{\pi}{6}\right)$

$= \left(-\cot\dfrac{\pi}{3}\right)\left(-\cot\dfrac{\pi}{6}\right)$

$= \dfrac{1}{\sqrt{3}} \times \sqrt{3} = 1$

11 sec A, sec B and sec C are positive in an acute angled triangle.

Also, arithmetic mean $\ge$ harmonic mean

$\Rightarrow \dfrac{\sec A+\sec B+\sec C}{3} \ge \dfrac{3}{\cos A+\cos B+\cos C}$

We have, in ΔABC,

$\cos A+\cos B+\cos C \le \dfrac{3}{2}$

$\Rightarrow \quad \dfrac{1}{\cos A+\cos B+\cos C} \ge \dfrac{2}{3}$

$\therefore \quad \dfrac{\sec A+\sec B+\sec C}{3} \ge 2$

$\Rightarrow \quad \sec A+\sec B+\sec C \ge 6$

12 Clearly, $(\sin A+\sin B)^2 + (\cos A+\cos B)^2 = 2$

$\therefore \quad 2+2(\sin A\sin B + \cos A\cos B) = 2$

$\Rightarrow \cos(B-A) = 0 \Rightarrow B = A+90°$

Second equation gives,

$\cos A - \sin A = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \cos(A+45°) = \cos 60°$

$\therefore \quad A = 15°$

13 Since, $\tan\dfrac{P}{2}+\tan\dfrac{Q}{2} = -\dfrac{b}{a}$

and $\tan\dfrac{P}{2}\tan\dfrac{Q}{2} = \dfrac{c}{a}$

Also, $\dfrac{P}{2}+\dfrac{Q}{2}+\dfrac{R}{2} = \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{P+Q}{2} = \dfrac{\pi}{4} \qquad \left[\because \angle R = \dfrac{\pi}{2}\right]$

$\Rightarrow \quad \dfrac{\tan\frac{P}{2}+\tan\frac{Q}{2}}{1-\tan\frac{P}{2}\tan\frac{Q}{2}} = 1 \Rightarrow \dfrac{-\frac{b}{a}}{1-\frac{c}{a}} = 1$

$\Rightarrow \quad c = a+b$

14 $\cos(\alpha+\beta) = \dfrac{4}{5}$

$\Rightarrow \quad \alpha+\beta \in$ 1st quadrant

$\sin(\alpha-\beta) = \dfrac{5}{13}$

$\Rightarrow \quad (\alpha-\beta) \in$ 1st quadrant

Now, as $\quad 2\alpha = (\alpha+\beta)+(\alpha-\beta)$

$\therefore \tan 2\alpha = \dfrac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$

$= \dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}} = \dfrac{56}{33}$

15 We have,

$z = \sin(\alpha+\beta) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad z^2 = x^2+y^2-2x^2y^2 + 2xy\sqrt{1-x^2}\sqrt{1-y^2}$

Now,

$\cos(\alpha+\beta) = \sqrt{1-x^2}\sqrt{1-y^2} - xy$

$= \dfrac{z^2-x^2-y^2+2x^2y^2}{2xy} - xy$

$= \dfrac{z^2-x^2-y^2}{2xy}$

16 Given that, $\sin\alpha+\sin\beta = -\dfrac{21}{65}$ ...(i)

and $\cos\alpha+\cos\beta = -\dfrac{27}{65}$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta + \cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta$

$= \left(-\dfrac{21}{65}\right)^2 + \left(-\dfrac{27}{65}\right)^2$

$\Rightarrow \quad 2+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) = \dfrac{441}{4225}+\dfrac{729}{4225}$

$\Rightarrow \quad 2[1+\cos(\alpha-\beta)] = \dfrac{1170}{4225}$

$\Rightarrow \quad \cos^2\left(\dfrac{\alpha-\beta}{2}\right) = \dfrac{1170}{4\times 4225}$

$\Rightarrow \quad \cos^2\left(\dfrac{\alpha-\beta}{2}\right) = \dfrac{9}{130}$

$\therefore \quad \cos\left(\dfrac{\alpha-\beta}{2}\right) = -\dfrac{3}{\sqrt{130}}$

$\left[\because \pi < \alpha-\beta < 3\pi \Rightarrow \dfrac{\pi}{2} < \dfrac{\alpha-\beta}{2} < \dfrac{3\pi}{2}\right]$

17 Since, $A+B+C = \pi$

$\therefore \quad A = \pi-(B+C)$

We have, $\cos A = \cos B\cos C$

$\Rightarrow \quad \cos[\pi-(B+C)] = \cos B\cos C$

$\Rightarrow \quad -\cos(B+C) = \cos B\cos C$

$\Rightarrow \quad -[\cos B\cos C - \sin B\sin C] = \cos B\cos C$

$\Rightarrow \quad \sin B\sin C = 2\cos B\cos C$

$\Rightarrow \quad \tan B\tan C = 2$

18 We know,

$\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\because \quad \tan\alpha = \dfrac{1}{1+2^{-x}}$

and $\quad \tan\beta = \dfrac{1}{1+2^{x+1}}$

$\tan(\alpha+\beta) = \dfrac{\frac{1}{1+\frac{1}{2^x}}+\frac{1}{1+2^{x+1}}}{1-\frac{1}{1+\frac{1}{2^x}}\cdot\frac{1}{1+2^{x+1}}}$

$\Rightarrow \tan(\alpha+\beta)$

$= \dfrac{2^x+2\cdot 2^{2x}+2^x+1}{1+2^x+2\cdot 2^x+2\cdot 2^{2x}-2^x}$

$\Rightarrow \tan(\alpha+\beta) = 1$

$\Rightarrow \quad \alpha+\beta = \dfrac{\pi}{4}$

19 We have, $\cos\left(x-\dfrac{\pi}{4}\right) = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \quad \tan\left(x-\dfrac{\pi}{4}\right) = \sqrt{7}$

$\Rightarrow \quad \dfrac{\tan x-1}{\tan x+1} = \sqrt{7}$

$\therefore \quad \tan x = \dfrac{\sqrt{7}+1}{1-\sqrt{7}} = \dfrac{-(4+\sqrt{7})}{3}$

20 $\operatorname{cosec} 10° - \sqrt{3}\sec 10°$

$= \dfrac{1}{\sin 10°} - \dfrac{\sqrt{3}}{\cos 10°}$

$= \left(\dfrac{\cos 10°}{2} - \dfrac{\sqrt{3}}{2}\sin 10°\right)\dfrac{4}{\sin 20°}$

$= \dfrac{4\sin(30°-10°)}{\sin 20°} = 4$

21 Now, $\sin 12°\sin 48°\sin 54°$

$= \dfrac{1}{2}(\cos 36° - \cos 60°)\cos 36°$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right]\left[\frac{\sqrt{5}+1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}-1}{4}\right]\left[\frac{\sqrt{5}+1}{4}\right]$$

$$= \frac{5-1}{32} = \frac{4}{32} = \frac{1}{8}$$

22 $1 + \cos 56° + \cos 58° - \cos 66°$

$$= 2\cos^2 28° + 2\sin 62° \sin 4°$$
$$= 2\cos^2 28° + 2\cos 28° \cos 86°$$
$$= 2\cos 28° (\cos 28° + \cos 86°)$$
$$= 2\cos 28° (2\cos 57° \cos 29°)$$
$$= 4\cos 28° \cos 29° \sin 33°$$

23 We have,

$$\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$$

On squaring both sides, we get

$$\sin^2\left(\frac{\pi}{2n}\right) + \cos^2\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{n}\right) = \frac{n}{4}$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{n}\right) = \frac{n}{4} - 1$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{n}\right) = \frac{n-4}{4}$$

$$\Rightarrow \quad n = 6 \text{ only}$$

24 $\tan A + 2\tan 2A + 4\tan 4A$

$$+ 8\left(\frac{1-\tan^2 4A}{2\tan 4A}\right)$$

$$= \tan A + 2\tan 2A + \left(\frac{4\tan^2 4A + 4 - 4\tan^2 4A}{\tan 4A}\right)$$

$$= \tan A + 2\tan 2A + 4\cot 4A$$

$$= \tan A + 2\tan 2A + 4\left(\frac{1-\tan^2 2A}{2\tan 2A}\right)$$

$$= \tan A + \left[\frac{2\tan^2 2A + 2 - 2\tan^2 2A}{\tan 2A}\right]$$

$$= \tan A + 2\cot 2A$$

$$= \tan A + 2\left(\frac{1-\tan^2 A}{2\tan A}\right)$$

$$= \frac{\tan^2 A + 1 - \tan^2 A}{\tan A} = \cot A$$

25 We have, $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{6}\right) + \frac{1}{\sqrt{2}}\cos\left(x + \frac{\pi}{6}\right)\right]$$

$$= \sqrt{2}\cos\left[x + \frac{\pi}{6} - \frac{\pi}{4}\right]$$

$$= \sqrt{2}\cos\left(x - \frac{\pi}{12}\right)$$

Hence, maximum value will be at $x = \frac{\pi}{12}$.

26 We have, $\cos A = m\cos B$

$$\Rightarrow \quad \frac{\cos A}{\cos B} = \frac{m}{1}$$

$$\Rightarrow \quad \frac{\cos A + \cos B}{\cos A - \cos B} = \frac{m+1}{m-1}$$

$$\Rightarrow \quad \frac{2\cos\frac{A+B}{2}\cos\frac{B-A}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}} = \frac{m+1}{m-1}$$

$$\Rightarrow \quad \cot\frac{A+B}{2} = \left(\frac{m+1}{m-1}\right)\tan\frac{B-A}{2}$$

But $\cot\frac{A+B}{2} = \lambda \tan\frac{B-A}{2}$

$$\therefore \quad \lambda = \frac{m+1}{m-1}$$

27 We have, $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right)$

$$= z\cos\left(\theta + \frac{4\pi}{3}\right) = k \text{ (say)}$$

$$\Rightarrow \quad \cos\theta = \frac{k}{x}, \quad \cos\left(\theta + \frac{2\pi}{3}\right) = \frac{k}{y}$$

and $\cos\left(\theta + \frac{4\pi}{3}\right) = \frac{k}{z}$

$$\therefore \quad \frac{k}{x} + \frac{k}{y} + \frac{k}{z} = \cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$= \cos\theta - \cos\left(\frac{\pi}{3} - \theta\right) - \cos\left(\frac{\pi}{3} + \theta\right)$$

$$= \cos\theta - 2\cos\frac{\pi}{3}\cos\theta$$

$$\Rightarrow \quad \cos\theta - 2 \times \frac{1}{2}\cos\theta = 0$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

28 $\cos^2\left(\frac{\pi}{3} - x\right) - \cos^2\left(\frac{\pi}{3} + x\right)$

$$= \left[\cos\left(\frac{\pi}{3} - x\right) + \cos\left(\frac{\pi}{3} + x\right)\right]\left[\cos\left(\frac{\pi}{3} - x\right) - \cos\left(\frac{\pi}{3} + x\right)\right]$$

$$= \left(2\cos\frac{\pi}{3}\cos x\right)\left(2\sin\frac{\pi}{3}\sin x\right)$$

$$= \sin\frac{2\pi}{3}\sin 2x = \frac{\sqrt{3}}{2}\sin 2x$$

Hence, maximum value of given expression is $\frac{\sqrt{3}}{2}$.

29 **Key idea** Apply the identify

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

and $\cos 3x = 4\cos^3 x - 3\cos x$

We have,

$$8\cos x\left[\cos\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2}\right] = 1$$

$$\Rightarrow \quad 8\cos x\left(\cos^2\frac{\pi}{6} - \sin^2 x - \frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{3}{4} - \sin^2 x - \frac{1}{2}\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{3}{4} - \frac{1}{2} - 1 + \cos^2 x\right) = 1$$

$$\Rightarrow \quad 8\cos x\left(\frac{-3 + 4\cos^2 x}{4}\right) = 1$$

$$\Rightarrow \quad 2(4\cos^3 x - 3\cos x) = 1$$

$$\Rightarrow \quad 2\cos 3x = 1 \Rightarrow \cos 3x = \frac{1}{2}$$

$$\Rightarrow \quad 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \quad [0 \le 3x \le 3\pi]$$

$$\Rightarrow \quad x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

Now, $\quad \text{Sum} = \frac{\pi}{9} + \frac{5\pi}{9} + \frac{7\pi}{9} = \frac{13\pi}{9}$

$$\Rightarrow \quad k\pi = \frac{13\pi}{9}$$

Hence, $\quad k = \frac{13}{9}$

30 $\because \quad \sin^2\theta \le 1$

$$\therefore \quad \frac{4xy}{(x+y)^2} \le 1$$

$$\Rightarrow \quad x^2 + y^2 + 2xy - 4xy \ge 0$$

$$\Rightarrow \quad (x-y)^2 \ge 0$$

which is true for all real values of x and y provided $x + y \ne 0$, otherwise $\frac{4xy}{(x+y)^2}$ will be meaningless.

31 Clearly, $p\sin x = \frac{\pi}{2} \pm p\cos x$

$$\Rightarrow \sin x \pm \cos x = \frac{\pi}{2p}$$

$$\Rightarrow \sin\left(x \pm \frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}p} \Rightarrow \left|\frac{\pi}{2\sqrt{2}p}\right| \le 1$$

For positive p, $p \ge \frac{\pi}{2\sqrt{2}}$ but $1 < \frac{\pi}{2\sqrt{2}} < 2$

Hence, the smallest positive integral value of p for which equation has solution is $p = 2$.

32 $A = \sin^2 x + \cos^4 x$

$$\Rightarrow \quad A = 1 - \cos^2 x + \cos^4 x$$

$= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4}$

$= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4}$ ... (i)

where, $0 \le \left(\cos^2 x - \frac{1}{2}\right)^2 \le \frac{1}{4}$ ... (ii)

∴ $\frac{3}{4} \le A \le 1$

33 $\cos 2\theta + 2\cos\theta = 2\cos^2\theta - 1 + 2\cos\theta$

$= 2\left(\cos\theta + \frac{1}{2}\right)^2 - \frac{3}{2} \ge -\frac{3}{2}$

$\left[\because 2\left(\cos\theta + \frac{1}{2}\right)^2 \ge 0, \forall\, \theta\right]$

and the maximum value of $\cos 2\theta + 2\cos\theta$ is 3.

34 Given that, $f(x) = \sin x - \sqrt{3}\cos x + 1$

∵ $-2 \le \sin x - \sqrt{3}\cos x \le 2$

$[\because \sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}]$

⇒ $-1 \le \sin x - \sqrt{3}\cos x + 1 \le 3$

∴ Range of $f(x) = [-1, 3]$

35 We have, $\sin^2 3x + \sin^4 x = 0$

$\Rightarrow \sin^2 x\{(3 - 4\sin^2 x)^2 + \sin^2 x\} = 0$

∴ $\sin x = 0 \Rightarrow x = n\pi$

Hence, greatest positive solution is 2π.

36 Clearly, $1 + \sin x \ge 0$

∴ The given equation becomes

$\cos x - \sin x = 1 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

⇒ $x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \ldots$

⇒ $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, \ldots$

∵ $0 \le x \le 3\pi$

∴ $x = 0, \frac{3\pi}{2}, 2\pi$

37 $\sin 2x + \cos x = 0$

⇒ $2\sin x\cos x + \cos x = 0$

$[\because \sin 2x = 2\sin x\cos x]$

⇒ $\cos x(2\sin x + 1) = 0$

⇒ $\cos x = 0$ or $\sin x = -\frac{1}{2}$

When $\cos x = 0$, then $x = (2n+1)\frac{\pi}{2}$

When $\sin x = -\frac{1}{2}$,

then $\sin x = -\sin\frac{\pi}{6}$

$\sin x = \sin\left(\pi + \frac{\pi}{6}\right)$

$[\because \sin(\pi + \theta) = -\sin\theta]$

$\sin x = \sin\frac{7\pi}{6}$

⇒ $x = n\pi + (-1)^n\frac{7\pi}{6}$ $[\because n \in Z]$

38 $\sin 2x - 2\cos x + 4\sin x = 4$

⇒ $(\sin x - 1)(2\cos x + 4) = 0$

⇒ $\sin x = 1, \cos x \ne -2$

∴ $x = n\pi + (-1)^n\frac{\pi}{2}$,

where $n \in z$

Hence, the required value of $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ in interval $[0, 5\pi]$.

39 We have,

$\sin\theta + \sin 4\theta + \sin 7\theta = 0$

⇒ $\sin 4\theta + (\sin\theta + \sin 7\theta) = 0$

⇒ $\sin 4\theta + 2\sin 4\theta \cdot \cos 3\theta = 0$

⇒ $\sin 4\theta\{1 + 2\cos 3\theta\} = 0$

⇒ $\sin 4\theta = 0, \cos 3\theta = -\frac{1}{2}$

As, $0 < \theta < \pi$

∴ $0 < 4\theta < 4\pi$

∴ $4\theta = \pi, 2\pi, 3\pi$

Also, $0 < 3\theta < 3\pi$

⇒ $3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

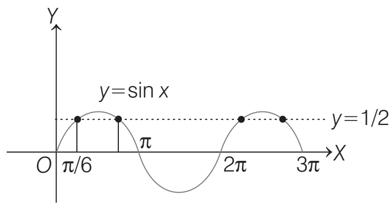
⇒ $\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

40 Given equation is

$2\sin^2 x + 5\sin x - 3 = 0.$

⇒ $(2\sin x - 1)(\sin x + 3) = 0$

⇒ $\sin x = \frac{1}{2}$ $[\because \sin x \ne -3]$



It is clear from figure that the curve intersect the line at four points in the given interval.

Hence, number of solutions are 4.

41 Since, α is a root of

$25\cos^2\theta + 5\cos\theta - 12 = 0$

∴ $25\cos^2\alpha + 5\cos\alpha - 12 = 0$

⇒ $(5\cos\alpha - 3)(5\cos\alpha + 4) = 0$

⇒ $\cos\alpha = -\frac{4}{5}$ and $\frac{3}{5}$

But $\frac{\pi}{2} < \alpha < \pi$ i.e. in second quadrant.

∴ $\cos\alpha = -\frac{4}{5}$

⇒ $\sin\alpha = \frac{3}{5}$

Now, $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \times \frac{3}{5} \times \left(-\frac{4}{5}\right) = -\frac{24}{25}$

42 $\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}$

$= \frac{\left(2\sin\frac{\pi}{7}\cdot\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\right)}{2\sin\frac{\pi}{7}}$

$= \frac{1}{8}\frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} = \frac{1}{8}\frac{\sin\left(\pi + \frac{\pi}{7}\right)}{\sin\frac{\pi}{7}} = -\frac{1}{8}$

$(\because 2\sin x\cos x = \sin 2x)$

Again, $\theta = \frac{\pi}{2^n - 1} \Rightarrow 2^n\theta = \pi + \theta$

∴ $\sin(2^n\theta) = -\sin\theta$

We know,

$\cos\theta\cdot\cos 2\theta\cdot\ldots\cdot\cos 2^{n-1}\theta$

$= \frac{1}{2^n}\frac{\sin(2^n\theta)}{\sin\theta}$

$= -\frac{1}{2^n}\frac{\sin\theta}{\sin\theta} = -\frac{1}{2^n}$

So, Statements I and II both are true and Statement II is a correct explanation for Statement I.

SESSION 2

1 Given, $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$,

where $x \in R$ and $k \ge 1$

Now, $f_4(x) - f_6(x)$

$= \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$

$= \frac{1}{4}(1 - 2\sin^2 x\cdot\cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x\cdot\cos^2 x)$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

2 From the given equations, we have

$\Sigma\tan\alpha = p$

$\Sigma\tan\alpha\tan\beta = 0$

and $\tan\alpha\tan\beta\tan\gamma = r$

Now,

$(1 + \tan^2\alpha)(1 + \tan^2\beta)(1 + \tan^2\gamma)$

$$= 1 + \Sigma \tan^2 \alpha + \Sigma \tan^2 \alpha \tan^2 \beta + \tan^2 \alpha \tan^2 \beta \tan^2 \gamma$$
$$= 1 + (\Sigma \tan \alpha)^2 - 2\Sigma \tan \alpha \tan \beta + (\Sigma \tan \alpha \tan \beta)^2 - 2 \tan \alpha \tan \beta \tan \gamma \Sigma \tan \alpha + \tan^2 \alpha \tan^2 \beta \tan^2 \gamma$$
$$= 1 + p^2 - 2pr + r^2 = 1 + (p - r)^2$$

3 $x = \dfrac{1}{1 - \cos^2 \phi} = \dfrac{1}{\sin^2 \phi}, y = \dfrac{1}{\cos^2 \phi}$

and $z = \dfrac{1}{1 - \sin^2 \phi \cdot \cos^2 \phi}$

Clearly, $\dfrac{1}{x} + \dfrac{1}{y} = 1$

$\Rightarrow$ $xy = x + y$ and $\dfrac{1}{z} = 1 - \dfrac{1}{xy}$

$\Rightarrow$ $xy = x + y$

and $xy = xyz - z$

$\therefore$ $xyz = xy + z = x + y + z$

4 The given equation is

$\sin^4 x + \cos^4 y + 2 = 4 \sin x \cos y$

$\Rightarrow$ $(\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2 \sin^2 x + 2 \cos^2 y - 4 \sin x \cos y = 0$

$\Rightarrow$ $(\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2 (\sin x - \cos y)^2 = 0$

which is possible only when

$\sin^2 x - 1 = 0, \cos^2 y - 1 = 0,$

$\sin x - \cos y = 0$

$\Rightarrow \sin^2 x = 1, \cos^2 y = 1, \sin x = \cos y$

As $0 \le x, y \le \dfrac{\pi}{2}$

$\therefore$ We get $\sin x = \cos y = 1$ and so

$\sin x + \cos y = 1 + 1 = 2$

5 Given equation is

$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$\Rightarrow (\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0$

$\Rightarrow$ $2 \cos 2x \cos x + 2 \cos 3x \cos x = 0$

$\Rightarrow$ $2 \cos x(\cos 2x + \cos 3x) = 0$

$\Rightarrow$ $2 \cos x\left(2 \cos \dfrac{5x}{2} \cos \dfrac{x}{2}\right) = 0$

$\Rightarrow$ $\cos x \cdot \cos \dfrac{5x}{2} \cdot \cos \dfrac{x}{2} = 0$

$\Rightarrow$ $\cos x = 0$ or $\cos \dfrac{5x}{2} = 0$ or $\cos \dfrac{x}{2} = 0$

Now, $\cos x = 0$

$\Rightarrow$ $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $[\because 0 \le x < 2\pi]$

$\cos \dfrac{5x}{2} = 0$

$\Rightarrow$ $\dfrac{5x}{2} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}, \dfrac{11\pi}{2}, \dots,$

$\Rightarrow$ $x = \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}$ $[\because 0 \le x < 2x]$

and $\cos \dfrac{x}{2} = 0$

$\Rightarrow$ $\dfrac{x}{2} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dots$

$\Rightarrow$ $x = \pi$ $[\because 0 \le x < 2\pi]$

Hence, $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \pi, \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}$

6 We have, $\sin (\theta + \alpha) = a$

and $\sin (\theta + \beta) = b$

$\Rightarrow \theta + \alpha = \sin^{-1} a$ and $\theta + \beta = \sin^{-1} b$

$\therefore$ $\alpha - \beta = \sin^{-1} a - \sin^{-1} b$

$= \dfrac{\pi}{2} - \cos^{-1} a - \dfrac{\pi}{2} + \cos^{-1} b$

$= \cos^{-1} b - \cos^{-1} a$

$= \cos^{-1} \{ab + \sqrt{1 - b^2} \sqrt{1 - a^2}\}$

$\Rightarrow$ $\cos (\alpha - \beta) = ab + \sqrt{(1 - a^2 - b^2 + a^2b^2)}$

$\Rightarrow$ $\cos^2 (\alpha - \beta) = a^2b^2 + 1 - a^2 - b^2 + a^2b^2 + 2ab\sqrt{(1 - a^2 - b^2 + a^2b^2)}$

Now, $\cos 2 (\alpha - \beta) - 4 ab \cos (\alpha - \beta)$

$= 2 \cos^2 (\alpha - \beta) - 1 - 4 ab \cos (\alpha - \beta)$

$= 4 a^2b^2 + 2 - 2a^2 - 2b^2 + 4ab\sqrt{1 - a^2 - b^2 + a^2b^2} - 1 - 4 a^2b^2 - 4ab\sqrt{1 - a^2 - b^2 + a^2b^2}$

$= 1 - 2a^2 - 2b^2$

7 Given, $\alpha = \sin^8 \theta + \cos^{14} \theta$

Since, $\sin^8 \theta \ge 0$ and $\cos^{14} \theta \ge 0$.

So, $\alpha \ge 0$

Also, $\sin^8 \theta + \cos^{14} \theta = 0$ is not possible.

Since, $\sin \theta = 0 \Rightarrow \cos \theta \ne 0$

and $\cos \theta = 0 \Rightarrow \sin \theta \ne 0$

So, $\alpha > 0$

Again, $\sin^2 \theta \le 1 \Rightarrow (\sin^2 \theta)^4 \le \sin^2 \theta$

$\Rightarrow$ $\sin^8 \theta \le \sin^2 \theta$

Also, $\cos^2 \theta \le 1$

$\Rightarrow$ $(\cos^2 \theta)^7 \le \cos^2 \theta$

$\Rightarrow$ $\cos^{14} \theta \le \cos^2 \theta$

So, $\alpha = \sin^8 \theta + \cos^{14} \theta \le \sin^2 \theta + \cos^2 \theta = 1$

$\alpha \le 1$ and $\alpha > 0$

8 Given, $\sin n\theta = \displaystyle\sum_{r=0}^{n} b_r \sin^r \theta = b_0 + b_1 \sin \theta + b_2 \sin^2 \theta + \dots + b_n \sin^n \theta$ ...(i)

Putting $\theta = 0$ in Eq. (i), we get $0 = b_0$

Again, Eq. (i) can be written as

$$\sin n\theta = \sum_{r=1}^{n} b_r \sin^r \theta$$
$$\frac{\sin n\theta}{\sin \theta} = \sum_{r=1}^{n} b_r \sin^{r-1} \theta$$

On taking limit as $\theta \to 0$, we get

$$\lim_{\theta \to 0} \frac{\sin n\theta}{\sin \theta} = b_1$$

$\Rightarrow$ $\displaystyle\lim_{\theta \to 0} n \left(\frac{\sin n\theta}{n\theta}\right) \left(\frac{\theta}{\sin \theta}\right) = b_1$

$\Rightarrow$ $n = b_1$

Hence, $b_0 = 0; b_1 = n$

9 Equation first can be written as

$x \sin a + y \times 2 \sin a \cos a + z \times \sin a (3 - 4 \sin^2 a)$

$= 2 \times 2 \sin a \cos a \cos 2a$

$\Rightarrow$ $x + 2y \cos a + z (3 + 4 \cos^2 a - 4)$

$= 4 \cos a (2 \cos^2 a - 1)$ as $\sin a \ne 0$

$\Rightarrow$ $8 \cos^3 a - 4z \cos^2 a - (2y + 4) \cos a + (z - x) = 0$

$\Rightarrow$ $\cos^3 a - \left(\dfrac{z}{2}\right)\cos^2 a - \left(\dfrac{y + 2}{4}\right)\cos a + \left(\dfrac{z - x}{8}\right) = 0$

which shows that $\cos a$ is a root of the equation

$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y + 2}{4}\right)t + \left(\frac{z - x}{8}\right) = 0$$

Similarly, from second and third equation we can verify that $\cos b$ and $\cos c$ are the roots of the given equation.

10 Given, $|\sin \theta . \cos \theta| + \sqrt{2 + \tan^2 \theta + \cot^2 \theta} = \sqrt{3}$

$\Rightarrow$ $|\sin \theta \cdot \cos \theta| + \sqrt{(\tan \theta + \cot \theta)^2} = \sqrt{3}$

$\Rightarrow$ $|\sin \theta \cdot \cos \theta| + |\tan \theta + \cot \theta| = \sqrt{3}$

$\Rightarrow$

$$|\sin \theta \cdot \cos \theta| + \left|\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cdot \cos \theta}\right| = \sqrt{3}$$

$\Rightarrow$ $|\sin \theta \cdot \cos \theta| + \dfrac{1}{|\sin \theta \cdot \cos \theta|} = \sqrt{3}$

We know that,

$$|\sin \theta \cdot \cos \theta| + \frac{1}{|\sin \theta \cdot \cos \theta|} \ge 2$$

Hence, there is no solution of this equation.

11 $(\sqrt{3})^{\sec^2\theta} = (\tan^2\theta + 1)^2 - 1$

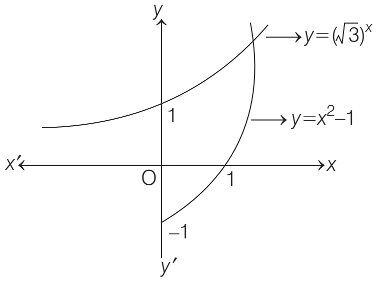
$= (\sec^2\theta)^2 - 1$

Put $\sec^2\theta = x$ $\quad (x \ge 1)$

Then, $(\sqrt{3})^x = x^2 - 1$

Let $y = (\sqrt{3})^x = (x^2 - 1)$ $\quad (x > 1)$

Now, graphs of $y = (\sqrt{3})^x$ and $y = x^2 - 1$ intersect at one point



i.e. $x = 2$, then $y = 3$

Thus, $\sec^2\theta = 2 \Rightarrow \sec\theta = \pm\sqrt{2}$

Therefore, there are two values of θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

12 Given equation is

$$e^{\sin x} - e^{-\sin x} = 4 \Rightarrow e^{\sin x} - \frac{1}{e^{\sin x}} = 4$$

Now, let $y = e^{\sin x}$

Then, we get

$$y - \frac{1}{y} = 4 \quad \Rightarrow \quad y^2 - 4y - 1 = 0$$

$$\therefore \quad y = \frac{4 \pm \sqrt{16 + 4}}{2} \Rightarrow y = 2 \pm \sqrt{5}$$

$$\Rightarrow \quad e^{\sin x} = 2 \pm \sqrt{5}$$

Since, sine is a bounded function i.e. $-1 \le \sin x \le 1$. Therefore, we get

$$e^{-1} \le e^{\sin x} \le e$$

$$\Rightarrow \quad e^{\sin x} \in \left[\frac{1}{e}, e\right]$$

Also, it is obvious that $2 + \sqrt{5} > e$

and $2 - \sqrt{5} < \frac{1}{e} \Rightarrow 2 \pm \sqrt{5} \notin \left[\frac{1}{e}, e\right]$

So, $e^{\sin x} = 2 + \sqrt{5}$ is not possible for any $x \in R$

and $e^{\sin x} = 2 - \sqrt{5}$ is also not possible for any $x \in R$

Hence, we can say that the given equation has no solution.

13 Given, $(\sqrt{3} - 1)\cos\theta + (\sqrt{3} + 1)\sin\theta = 2$ ...(i)

Let $(\sqrt{3} - 1) = r\cos\alpha$

and $(\sqrt{3} + 1) = r\sin\alpha$

Then, $r^2(\cos^2\alpha + \sin^2\alpha) = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2$

$$\Rightarrow \quad r^2 = 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$$

$$\Rightarrow \quad r^2 = 8 \Rightarrow r = 2\sqrt{2}$$

and $\frac{r\sin\alpha}{r\cos\alpha} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = -\frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$

$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \tan\alpha = \tan\left(\frac{5\pi}{12}\right) \Rightarrow \alpha = \frac{5\pi}{12}$$

Also, Eq. (i) we have,

$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = 2$

$$\Rightarrow \quad 2\sqrt{2}\cos(\alpha - \theta) = 2$$

$$\Rightarrow \quad \cos(\theta - \alpha) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta - \alpha = 2n\pi \pm \frac{\pi}{4}$$

$$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{4} + \frac{5\pi}{12}$$

14 Given, $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$

[For intersection X –axis, put $y = 0$]

$$\Rightarrow \quad 81^{\sin^2 x} + 81^{1 - \sin^2 x} - 30 = 0$$

$$\Rightarrow \quad 81^{2\sin^2 x} + 81 - 30 \cdot 81^{\sin^2 x} = 0$$

[multiplying by $81^{\sin^2 x}$]

$$\Rightarrow \quad 81^{2(\sin^2 x)} - 3 \cdot 81^{\sin^2 x} - 27 \cdot 81^{\sin^2 x} + 81 = 0$$

$$\Rightarrow \quad (81^{\sin^2 x} - 3)(81^{\sin^2 x} - 27) = 0$$

$$\Rightarrow \quad [(3)^4]^{\sin^2 x} = (3)^1 \text{ or } [(3^4)]^{\sin^2 x} = (3)^3$$

$$\Rightarrow \quad \sin x = \pm\frac{1}{2} \text{ or } \sin x = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6} \text{ or } x = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$$

Clearly, the graph of $y = 81^{\sin^2 x} + 81^{\cos^2 x} - 30$ intersects the X-axis at eight points in $-\pi \le x \le \pi$.

15 **Given** A ΔPQR such that

$3\sin P + 4\cos Q = 6$ ... (i)

$4\sin Q + 3\cos P = 1$ ... (ii)

On squaring and adding the Eqs. (i) and (ii), we get

$$(3\sin P + 4\cos Q)^2 + (4\sin Q + 3\cos P)^2 = 36 + 1$$

$$\Rightarrow \quad 9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q) + 2 \times 3 \times 4(\sin P\cos Q + \sin Q\cos P) = 37$$

$$\Rightarrow \quad 24[\sin(P + Q)] = 37 - 25$$

$$\Rightarrow \quad \sin(P + Q) = \frac{1}{2}$$

Since, P and Q are angles of ΔPQR, therefore,

$$0° < P, Q < 180°$$

$$\Rightarrow \quad P + Q = 30° \text{ or } 150°$$

$$\Rightarrow \quad R = 150° \text{ or } 30°$$

Hence, two cases aries here.

Case I When, $R = 150°$

$$R = 150° \Rightarrow P + Q = 30°$$

$$\Rightarrow \quad 0 < P, Q < 30°$$

$$\Rightarrow \quad \sin P < \frac{1}{2}, \cos Q < 1$$

$$\Rightarrow \quad 3\sin P + 4\cos Q < \frac{3}{2} + 4$$

$$\Rightarrow \quad 3\sin P + 4\cos Q < \frac{11}{2} < 6$$

$$\Rightarrow \quad 3\sin P + 4\cos Q < 6 \text{ not possible}$$

Case II $R = 30°$

Hence, $R = 30°$ is the only possibility.